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Canonical analysis of time series

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Summary

Selected generalizations of classical canonical analysis to canonical analysis of multivariate autoregressive time series are presented. It is shown how to construct the autocovariance matrices and the cross-covariance matrices under the assumption that the parameters of the autoregressive processes are known. In order to be able to construct the canonical series based on the sample the estimation method of the parameters and the rank of the multivariate autoregressive model is given. The theoretical considerations are illustrated by a numerical example.

1. Introduction

The multiple regression is a useful tool in investigation of the dependence between one dependent variable and a set of p independent variables. Very often, however, we are interested in more complicated form of dependence, namely dependence between a set of q dependent variables and a set of p independent variables. Hotelling (1936) introduced the idea of the canonical variables and canonical correlations and suggested using them in the investigation of the dependence of two vectors of variables. The ideas of Hotelling have been generalized in many papers and found many applications (see e.g. Krzyśko, 1982; Gittins 1985).

This paper contains selected generalizations of classical canonical analysis to canonical analysis of multivariate autoregressive time series. The dependences between linear combinations of the processes $X(t)$ and $Y(t)$ are investigated. As the measure of the association between the linear combinations of the processes $X(t)$ and $Y(t)$ we can use the cross-covariance coefficient that depends on autocovariances and cross covariances matrices of the processes. It is shown how to construct the autocovariance matrices and the matrix of cross covariances under the assumption that the parameters of the autoregressive processes are known. In order to make possible the construction of the sample canonical series of the autoregressive

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Key words: Canonical correlations, canonical series, autoregressive processes

processes the methods of estimation of the parameters and the order of the model are showed. The theoretical considerations are illustrated by a numerical example.

2. Canonical series and canonical correlations

Let $\mathbf{Z}(t) = (\mathbf{X}'(t), \mathbf{Y}'(t))' = (X_1(t),..., X_p(t), Y_1(t),...,Y_q(t))'$, $t=0, \pm 1, \pm 2,...$, be a $(p+q)$-dimensional jointly strictly stationary real valued vector process with zero mean vector and the positive definite autocovariance matrix

$$\mathbf{C}_\mathbf{Z}(\tau) = E[\mathbf{Z}(t)\mathbf{Z}'(t+\tau)] = \begin{bmatrix} \mathbf{C}_\mathbf{X}(\tau), & \mathbf{C}_\mathbf{XY}(\tau) \\ \mathbf{C}_\mathbf{YX}(\tau), & \mathbf{C}_\mathbf{Y}(\tau) \end{bmatrix},$$

where $\mathbf{C}'_\mathbf{YX}(\tau) = \mathbf{C}_\mathbf{XY}(-\tau)$. Suppose now that $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are autoregressive processes.

Let $\{\mathbf{X}(t),\ t=0, \pm 1, \pm 2,...\}$ be a p-dimensional stochastic process described by an autoregressive equation of finite order r_1

$$\sum_{u=0}^{r_1} \mathbf{A}(u)\, \mathbf{X}(t-u) = \boldsymbol{\psi}_1(t), \tag{1}$$

where

(i) $\mathbf{X}(t)$ are observable random vectors,

(ii) $\mathbf{A}(u)$ are matrices of parameters such that the absolute value of roots of the characteristic equation

$$\left| \sum_{u=0}^{r_1} \mathbf{A}(u)\, \lambda^{r_1-u} \right| = 0$$

are less than 1,

(iii) $\boldsymbol{\psi}_1(t)$ are unobservable independently normally distributed random vectors with zero mean and positive definite covariance matrix $\boldsymbol{\Sigma}_1$.

Suppose that the parameters $\mathbf{A}(1), \mathbf{A}(2), ..., \mathbf{A}(r_1), \boldsymbol{\Sigma}_1$ are known.

We shall express the covariance matrix $\mathbf{C}_\mathbf{X}(\tau)$ by the parameters of the autoregressive equation.

The process $\mathbf{X}(t)$ as a strictly stationary process can be expressed in the unique infinite moving average representation

$$\mathbf{X}(t) = \sum_{u=0}^{\infty} \mathbf{U}_1(u)\, \boldsymbol{\psi}_1(t-u), \qquad \mathbf{U}_1(0) = \mathbf{I}_p.$$

It is clear that this process can also be expressed as a first order vector autoregressive process

$$\boldsymbol{\zeta}(t) = \mathbf{A}\, \boldsymbol{\zeta}(t-1) + \boldsymbol{\varepsilon}_1(t),$$

where

$$\zeta(t) = (X'(t), ..., X'(t-r_1+1))',$$

$$\varepsilon_1(t) = (\psi_1'(t), 0', ..., 0')'$$

and

$$\mathbf{A} = \begin{bmatrix} -\mathbf{A}(1), & -\mathbf{A}(2), ..., -\mathbf{A}(r_1) \\ \mathbf{I}_{p(r_1-1)} & , \quad \mathbf{0} \end{bmatrix}.$$

Note that

$$\mathbf{X}(t) = \mathbf{W}_1'\zeta(t), \quad \psi_1(t) = \mathbf{W}_1'\varepsilon_1(t) \text{ and } \varepsilon_1(t) = \mathbf{W}_1\psi_1(t),$$

where $\mathbf{W}_1' = (\mathbf{I}_p, 0, \ldots, 0)$ is $p \times pr_1$ matrix. Also, $\zeta(t)$ has the representation

$$\zeta(t) = \sum_{u=0}^{\infty} \mathbf{A}^u \varepsilon_1(t-u)$$

so that

$$\mathbf{X}(t) = \sum_{u=0}^{\infty} \mathbf{W}_1' \mathbf{A}^u \mathbf{W}_1 \psi_1(t-u).$$

Thus, from the unique moving average representation of $\mathbf{X}(t)$ we have

$$\mathbf{U}_1(u) = \mathbf{W}_1' \mathbf{A}^u \mathbf{W}_1, \quad u = 0, 1, \ldots.$$

Hence

$$\mathbf{C_X}(\tau) = E[\mathbf{X}(t)\mathbf{X}'(t+\tau)] = \sum_{u=0}^{\infty} \mathbf{U}_1(u)\, \Sigma_1\, \mathbf{U}_1'(u+\tau).$$

The spectral decomposition of a matrix $\mathbf{A}$ has the form

$$\mathbf{A} = \mathbf{P}_1 \Lambda_1 \mathbf{P}_1^{-1},$$

where

$$\Lambda_1 = \text{diag}(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1,pr_1}).$$

Hence

$$\mathbf{A}^u = \mathbf{P}_1 \Lambda_1^u \mathbf{P}_1^{-1}$$

and

$$\mathbf{C_X}(\tau) = \mathbf{W}_1' \mathbf{P}_1 \mathbf{H}_X^0(\tau) \mathbf{P}_1' \mathbf{W}_1,$$

where

$$\mathbf{H_X} = (h_{X,mn}) = \mathbf{P}_1^{-1} \mathbf{W}_1 \Sigma_1 \mathbf{W}_1' (\mathbf{P}_1^{-1})',$$

$$\mathbf{H}_{\mathbf{X}}^{0}(\tau)=\sum_{u=0}^{\infty}\boldsymbol{\Lambda}_{1}^{u}\,\mathbf{H}_{\mathbf{X}}\boldsymbol{\Lambda}_{1}^{u+\tau}.$$

The eigenvalues of the matrix **A** are equal to the roots of the characteristic equation

$$|\sum_{u=0}^{r_1}\mathbf{A}(u)\,\lambda^{r_1-u}|=0$$

which absolute values are less than 1. Hence

$$\sum_{u=0}^{\infty}\boldsymbol{\Lambda}_{1}^{u}\mathbf{H}_{X}\,\boldsymbol{\Lambda}_{1}^{u+\tau}<\infty$$

and the elements of the matrix $\mathbf{H}_{\mathbf{X}}^{0}(\tau)$ are

$$h_{\mathbf{X},mn}^{0}(\tau)=h_{\mathbf{X},mn}\,\lambda_{1n}^{\tau}\,(1-\lambda_{1m}\,\lambda_{1n})^{-1}.$$

Let $\{\mathbf{Y}(t),\quad t=0,\pm1,\pm2,\ldots\}$ be a q-dimensional stochastic process described by an autoregressive equation of finite order r_2

$$\sum_{u=0}^{r_2}\mathbf{B}(u)\,\mathbf{Y}(t-u)=\boldsymbol{\psi}_2(t). \tag{2}$$

By following the same arguments as in the case of a process $\mathbf{X}(t)$ we have

$$\mathbf{C}_{\mathbf{Y}}(\tau)=\mathbf{W}_2'\,\mathbf{P}_2\,\mathbf{H}_{\mathbf{Y}}^{0}(\tau)\,\mathbf{P}_2'\,\mathbf{W}_2\ ,$$

where

$$\mathbf{H}_{\mathbf{Y}}=(h_{\mathbf{Y},mn})=\mathbf{P}_2^{-1}\,\mathbf{W}_2\,\boldsymbol{\Sigma}_2\,\mathbf{W}_2'\,\mathbf{P}_2'^{-1},$$

$$\mathbf{H}_{\mathbf{Y}}^{0}(\tau)=(h_{\mathbf{Y},mn}^{0}(\tau))$$

$$h_{\mathbf{Y},mn}^{0}(\tau)=h_{\mathbf{Y},mn}\,\lambda_{2n}^{\tau}\,(1-\lambda_{2m}\,\lambda_{2n})^{-1},$$

$$\mathbf{W}_2'=(\mathbf{I}_q,\mathbf{0},\ldots,\mathbf{0})$$

and λ_{2j} are the eigenvalues of the matrix

$$\mathbf{B}=\begin{bmatrix}-\mathbf{B}(1), & -\mathbf{B}(2), \ldots & -\mathbf{B}(r_2)\\ \mathbf{I}_{q(r_2-1)} & & \mathbf{0}\end{bmatrix}.$$

Let

$$E\{\boldsymbol{\psi}_1(r)\,\boldsymbol{\psi}_2(s)\}=\begin{cases}\boldsymbol{\Sigma}_{12}, & \text{for } r=s\\ \mathbf{0}, & \text{for } r\neq s\end{cases}.$$

Then the cross-covariance matrix between $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ has the following form

$$\mathbf{C}_{\mathbf{XY}}(\tau)=E[\mathbf{X}(t)\mathbf{Y}'(t+\tau)]=\mathbf{W}_1'\,\mathbf{P}_1\mathbf{H}_{\mathbf{XY}}^{0}(\tau)\,\mathbf{P}_2'\mathbf{W}_2\ ,$$

where

$$\mathbf{H_{XY}} = (h_{\mathbf{XY},m,n}) = \mathbf{P}_1^{-1}\mathbf{W}_1\,\mathbf{\Sigma}_{12}\mathbf{W}_2'\,(\mathbf{P}_2^{-1})',$$

$$\mathbf{H}^0_{\mathbf{XY}}(\tau) = (h_{\mathbf{XY},m,n}(\tau))\,,$$

$$h^0_{\mathbf{XY},m,n}(\tau) = h_{\mathbf{XY},m,n}\,\lambda_{2n}^{\tau}\frac{1}{1-\lambda_{1m}\,\lambda_{2n}}\,.$$

Consider now an arbitrary linear combination, $V(t) = \mathbf{L}'\mathbf{X}(t)$, of the components of $\mathbf{X}(t)$ and an arbitrary linear function, $V(t) = \mathbf{M}'\mathbf{Y}(t)$, of the components of $\mathbf{Y}(t)$.

We first ask for the linear functions that have maximum cross correlation $\rho_{UV}(0)$.

By definition,

$$\rho_{UV}(0) = \frac{\mathbf{C}_{UV}(0)}{\sqrt{\mathbf{C}_U(0)\mathbf{C}_V(0)}}\,,$$

where

$$\mathbf{C}_U(0) = \mathbf{L}'\mathbf{C_X}(0)\mathbf{L}\,,$$

$$\mathbf{C}_V(0) = \mathbf{M}'\mathbf{C_Y}(0)\mathbf{M}\,,$$

$$\mathbf{C}_{UV}(0) = \mathbf{L}'\mathbf{C_{XY}}(0)\mathbf{M}\,.$$

Hence

$$\rho_{UV}(0) = \frac{\mathbf{L}'\mathbf{C_{XY}}(0)\mathbf{M}}{\sqrt{\mathbf{L}'\mathbf{C_X}(0)\mathbf{L}\cdot\mathbf{M}'\mathbf{C_Y}(0)\mathbf{M}}}\,.$$

Since the cross-correlation of a multiple of $U(t)$ and a multiple of $V(t)$ is the same as the cross-correlation of $U(t)$ and $V(t)$, we can make an arbitrary normalization of $\mathbf{L}$ and $\mathbf{M}$. We therefore require $\mathbf{L}$ and $\mathbf{M}$ to be such that $U(t)$ and $V(t)$ have unit variance, that is

$$\mathbf{C}_U(0) = \mathbf{L}'\mathbf{C_X}(0)\mathbf{L} = 1\,, \tag{3}$$

$$\mathbf{C}_V(0) = \mathbf{M}'\mathbf{C_Y}(0)\mathbf{M} = 1\,. \tag{4}$$

Then the cross-correlation between $U(t)$ and $V(t)$ is

$$\rho_{UV}(0) = \mathbf{L}'\mathbf{C_{XY}}(0)\mathbf{M}\,. \tag{5}$$

Thus the algebraic problem is to find $\mathbf{L}$ and $\mathbf{M}$ which maximize (5) subject to (3) and (4).

Let

$$F(\mathbf{L},\mathbf{M}) = \mathbf{L}'\mathbf{C_{XY}}(0)\mathbf{M} - \lambda/2(\mathbf{L}'\mathbf{C_X}(0)\mathbf{L} - 1) - \mu/2(\mathbf{M}'\mathbf{C_Y}(0)\mathbf{M} - 1)\,, \tag{6}$$

where λ and μ are Lagrange multipliers. We differentiate F with respect to the elements of $\mathbf{L}$ and $\mathbf{M}$. The vectors of derivatives set equal to zero are

$$\frac{\partial F}{\partial L} = \mathbf{C_{XY}}(0)\mathbf{M} - \lambda\mathbf{C}_X(0)\mathbf{L} = \mathbf{0}\,, \tag{7}$$

$$\frac{\partial F}{\partial M} = \mathbf{C}'_{\mathbf{XY}}(0)\mathbf{L} - \mu \mathbf{C}_{\mathbf{Y}}(0)\mathbf{M} = 0 \, . \tag{8}$$

Multiplication of (7) on the left by $\mathbf{L}'$ and (8) on the left by $\mathbf{M}'$ gives

$$\mathbf{L}'\mathbf{C}_{\mathbf{XY}}(0)\mathbf{M} - \lambda \mathbf{L}'\mathbf{C}_{\mathbf{X}}(0)\mathbf{L} = 0 \, ,$$

$$\mathbf{M}'\mathbf{C}_{\mathbf{XY}}(0)\mathbf{L} - \mu \mathbf{M}'\mathbf{C}_{\mathbf{Y}}(0)\mathbf{M} = 0 \, .$$

Since $\mathbf{L}'\,\mathbf{C}_{\mathbf{X}}(0)\mathbf{L} = 1$ and $\mathbf{M}'\,\mathbf{C}_{\mathbf{Y}}(0)\mathbf{M} = 1$, this shows that $\lambda = \mu = \mathbf{L}'\,\mathbf{C}_{\mathbf{XY}}(0)\,\mathbf{M} = \rho$. Thus (7) and (8) can be written as

$$-\rho \mathbf{C}_{\mathbf{X}}(0)\mathbf{L} + \mathbf{C}_{\mathbf{XY}}(0)\mathbf{M} = 0 \, ,$$

$$\mathbf{C}_{\mathbf{YX}}(0)\mathbf{L} - \rho \mathbf{C}_{\mathbf{Y}}(0)\mathbf{M} = 0 \, ,$$

since

$$\mathbf{C}'_{\mathbf{XY}}(0) = \mathbf{C}_{\mathbf{YX}}(0).$$

In one matrix equation this is

$$\begin{bmatrix} -\rho \mathbf{C}_{\mathbf{X}}(0) \, , & \mathbf{C}_{\mathbf{XY}}(0) \\ \mathbf{C}_{\mathbf{YX}}(0) \, , & -\rho \mathbf{C}_{\mathbf{Y}}(0) \end{bmatrix} \begin{bmatrix} \mathbf{L} \\ \mathbf{M} \end{bmatrix} = 0 \, .$$

In order that there be a nontrivial solution, the matrix on the left must be singular, that is,

$$\begin{vmatrix} -\rho \mathbf{C}_{\mathbf{X}}(0) \, , & \mathbf{C}_{\mathbf{XY}}(0) \\ \mathbf{C}_{\mathbf{YX}}(0) \, , & -\rho \mathbf{C}_{\mathbf{Y}}(0) \end{vmatrix} = 0 \, .$$

We can express this condition in one of two alternative ways:

$$| \mathbf{C}_{\mathbf{XY}}(0)\mathbf{C}_{\mathbf{Y}}^{-1}(0)\mathbf{C}_{\mathbf{YX}}(0) - \rho^2 \, \mathbf{C}_{\mathbf{X}}(0) \, | = 0 \tag{9}$$

or

$$| \mathbf{C}_{\mathbf{YX}}(0)\mathbf{C}_{\mathbf{X}}^{-1}(0)\mathbf{C}_{\mathbf{XY}}(0) - \rho^2 \, \mathbf{C}_{\mathbf{Y}}(0) \, | = 0 \, . \tag{10}$$

Equation (9) has p roots, while equation (10) has q. The nonzero roots of equations (9) and (10) are equal, so that it is possible to represent them with the same symbols. The number of nonzero roots of these equations is equal to the rank of the matrix $\mathbf{C}_{\mathbf{XY}}(0)$.

Let $\rho_1^2 \geq \rho_2^2 \geq \ldots \geq \rho_p^2$ be the roots and $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_p$ the corresponding vectors of the determinantal equation (9) and let $\rho_1^2 \geq \rho_2^2 \geq \ldots \geq \rho_q^2$ be the roots and $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_q$ the corresponding vectors of the determinantal equation (10). Let $L = (\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_p)$ and $M = (\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_q)$.

The non-zero roots $\rho_1, \rho_2, \ldots, \rho_s$, where $s = \text{rank } \mathbf{C}_{\mathbf{XY}}(0)$, are called the canonical correlations.

The linear functions $U(t) = L'\mathbf{X}(t)$ are called the canonical series of the process $\mathbf{X}(t)$ and the linear functions $V(t) = M'\mathbf{Y}(t)$ are called the canonical series of the process $\mathbf{Y}(t)$.

3. Properties of canonical series

Let

$$\underset{p\times q}{\mathbf{P}} = \left[\begin{array}{cccc|c} \rho_1 & & & & \\ & \rho_2 & & & \\ & & \cdot & & \mathbf{0} \\ & & & \cdot & \\ & & & \rho_s & \\ \hline & & \mathbf{0} & & \mathbf{0} \end{array}\right].$$

(i) The autocovariance matrix of the canonical series has the following form

$$\mathrm{Var}\begin{bmatrix} U(t) \\ V(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p\,, & \mathbf{P} \\ \mathbf{P}\,, & \mathbf{I}_q \end{bmatrix}. \tag{11}$$

From (11) it can be noted that the variances of all canonical series $U_i(t)$ and $V_i(t)$ are equal to unity. Also

$$\mathrm{corr}\,(U_i(t)\,,\,V_j(t)) = \begin{cases} \rho_i\,, & \text{if } i=j\,, \\ 0\,, & \text{if } i\neq j\,, \end{cases}$$

and

$$\mathrm{corr}\,(U_i(t),\,U_j(t)) = 0,\quad \mathrm{corr}\,(V_i(t),\,V_j(t)) = 0,$$

if $i \neq j$.

In other words, the first canonical series of the process $\mathbf{X}(t)$ is only correlated with the first canonical series of the process $\mathbf{Y}(t)$, the second with the second, and so on. These correlations are respectively equal $\rho_1,\ldots,\rho_s$.

(ii) The following relationships are true:

$$\mathrm{corr}\,(X_i(t),\,V_j(t)) = \rho_j\,\mathrm{corr}\,(X_i(t),U_j(t)),$$
$$\text{for } i = 1,2,\ldots,p\,,\quad j = 1,2,\ldots,s\,,$$

$$\mathrm{corr}\,(Y_i(t),\,U_j(t)) = \rho_j\,\mathrm{corr}\,(Y_i(t),V_j(t)),$$
$$\text{for } i = 1,2,\ldots,q\,,\quad j = 1,2,\ldots,s.$$

Also

$$\sum_{j=1}^{p} \mathrm{corr}^2(X_i(t),\,U_j(t)) = 1\,,$$

$$\sum_{j=1}^{q} \operatorname{corr}^2(Y_k(t), V_j(t)) = 1 ,$$

for $i = 1,2,...,p,\ \ k = 1,2,..,q$

or

$$\operatorname{Var} X_i(t) = \sum_{j=1}^{p} \operatorname{Cov}^2(X_i(t), U_j(t)),$$

$$\operatorname{Var} Y_k(t) = \sum_{j=1}^{q} \operatorname{Cov}^2(Y_k(t), V_j(t)),$$

for $i = 1,2,...,p,\ \ k = 1,2,...,q.$

The above relationships form the basis for the interpretation of the canonical series. It is obvious from these relationships that individual canonical series account, in varying degrees, for the variance of the original process. A measure of this explanation is the coefficient of determination between the canonical series and the original process.

(iii) The canonical correlations and canonical series are invariant with respect to nonsingular linear transformations.

4. Parameters estimation and order determination of the multivariate autoregressive processes

In practice the parameters and orders of the multivariate autoregressive processes are not known. We first assume that the order r_1 of (1) is known and that the parameters $A(1),...,A(r_1),\Sigma_1$ are not known. We will now consider the estimation of the parameters $A(1),...,A(r_1)$ on the basis of N independent realizations $\mathbf{X}_n(1),...,\mathbf{X}_n(T)$, $n=1,...,N$ of the process $\{\mathbf{X}(t)\}$. If we introduce the following notation

$$\underset{p\times pr_1}{A} = (A(1),...,A(r_1)), \quad \underset{pr_1\times 1}{\zeta_n(t-1)} = (\mathbf{X}'_n(t-1),...,\mathbf{X}'_n(t-r_1))' ,$$

$$\underset{p\times N(T-r_1)}{\mathbf{X}} = (\mathbf{X}_1(r_1+1),...,\mathbf{X}_1(T),...,\mathbf{X}_N(r_1+1),...,\mathbf{X}_N(T)) ,$$

$$\underset{p\times N(T-r_1)}{\zeta} = (\zeta_1(r_1),...,\zeta_1(T-1),...,\zeta_N(r_1),...,\zeta_N(T-1)) ,$$

$$\underset{p\times N(T-r_1)}{\boldsymbol{\psi}_1} = (\boldsymbol{\psi}_{11}(r_1+1),...,\boldsymbol{\psi}_{11}(T),...,\boldsymbol{\psi}_{1N}(r_1+1),...,\boldsymbol{\psi}_{1N}(T)) ,$$

then model (1) for $t = r_1+1,...,T$, $n = 1,...,N$ may be rewritten as

$$\mathbf{X} = -A\,\zeta + \boldsymbol{\psi}_1 . \tag{12}$$

For $N(T-r_1) \geq pr_1$, the ordinary least squares estimator of A is (Krzyśko and Smoczyński (1984)):

$$\hat{A} = -\mathbf{X}\zeta'(\zeta\zeta')^{-1} = -\left[\sum_{n=1}^{N} \sum_{t=r_1+1}^{T} \mathbf{X}_n(t)\, \zeta'_n(t-1)\right]\mathbf{D}_1^{-1}, \tag{13}$$

where

$$\mathbf{D}_1 = \sum_{n=1}^{N} \sum_{t=r_1+1}^{T} \zeta_n(t-1)\, \zeta'_n(t-1)\ .$$

From (12) we have

$$\hat{\boldsymbol{\psi}}_1 = \mathbf{X} + \hat{A}\,\zeta$$

and

$$\hat{\boldsymbol{\Sigma}}_1 = [N(T-r_1)]^{-1}\hat{\boldsymbol{\psi}}_1\hat{\boldsymbol{\psi}}_1' = [N(T-r_1)]^{-1}(\mathbf{X}\mathbf{X}' + \mathbf{X}\,\zeta'\hat{A}')\ . \tag{14}$$

Model (2) for $t = r_2+1,\ldots,T,\ n = 1,\ldots,N$ may be rewritten as

$$\mathbf{Y} = -B\,\zeta + \boldsymbol{\psi}_2\ . \tag{15}$$

For $N(T-r_2) \geq qr_2$, the ordinary least squares estimator of B is

$$\hat{B} = -\mathbf{Y}\,\zeta'(\zeta\,\zeta')^{-1} = -\left[\sum_{n=1}^{N} \sum_{t=r_2+1}^{T} \mathbf{Y}_n(t)\, \zeta'_n(t-1)\right]\mathbf{D}_2^{-1}\ , \tag{16}$$

where

$$\mathbf{D}_2 = \sum_{n=1}^{N} \sum_{t=r_2+1}^{T} \zeta_n(t-1)\, \zeta'(t-1)\ .$$

From (15) we have

$$\hat{\boldsymbol{\psi}}_2 = \mathbf{X} + \hat{A}\,\zeta$$

and

$$\hat{\boldsymbol{\Sigma}}_2 = [N(T-r_2)]^{-1}\,\hat{\boldsymbol{\psi}}_2\,\hat{\boldsymbol{\psi}}_2' = [N(T-r_2)]^{-1}\,(\mathbf{Y}\mathbf{Y}' + \mathbf{Y}\,\zeta'\hat{\mathbf{B}}')\ . \tag{17}$$

Suppose now that $r_1 < r_2$. Then the estimator of the matrix $\boldsymbol{\Sigma}_{12}$ has the following form

$$\hat{\boldsymbol{\Sigma}}_{12} = [N(T-r_1)]^{-1}\hat{\boldsymbol{\psi}}_1\hat{\boldsymbol{\psi}}_2', \tag{18}$$

where the matrix $\hat{\boldsymbol{\psi}}_2$ contains the $N(T-r_1)$ first columns of the matrix $\hat{\boldsymbol{\psi}}_2$.

In practice, not only are the parameters of the models (1) and (2) unknown but so are the orders of the models. Therefore, we will now consider the estimation of the true order r_1 of the autoregressive equation (1) on the basis of N independent realizations $\mathbf{X}_n(1),\ldots,\mathbf{X}_n(T),\ n = 1,\ldots,N$ of the process $\{\mathbf{X}(t)\}$. Quinn (1980) proposed an expression

$\varphi(k)$ for the determination of the order of multivariate autoregressive models, of the following form:

$$\varphi(k) = \ln|\hat{\mathbf{\Sigma}}_1^{(k)}| + 2T^{-1}p^2 k \ln\ln T \ , \tag{19}$$

where $\hat{\mathbf{\Sigma}}_1^{(k)}$ is given by (14) in the model of the order k.

The estimator $\hat{r}_1$ of the true order of the model (1) can be expressed as

$$\hat{r}_1 = \arg\{ \min_{k \in \{1,\dots,R_1\}} \varphi(k) \},$$

where R_1 is an arbitrarily chosen number larger that r_1.

Quinn (1970) shows that the estimator $\hat{r}_1$ is a strongly consistent estimator of the true order r_1.

As the true order of the model (1) is not known, we can assume that $r_1 \in \{1,\dots,R_1\}$. For each assumed order we can obtain the estimators of the remaining parameters of the model (1) by (13) and (14). In this manner we can compute the values of the function (19) for each $k \in \{1,\dots,R_1\}$, and we can find the estimator of the order of the model (1) by minimizing the function $\varphi(k)$.

In the same manner we can find the estimator of the true order of the model (2).

5. Example

Consider the second order two-dimensional autoregressive process of the form

$$\mathbf{X}(t) + \begin{bmatrix} -1.0, & -0.3 \\ 3.3, & 1.0 \end{bmatrix} \mathbf{X}(t-1) + \begin{bmatrix} -0.02, & 0 \\ 0\,, & -0.02 \end{bmatrix} \mathbf{X}(t-2) = \boldsymbol{\psi}_1(t),$$

where

$$\boldsymbol{\psi}_1(t) \sim N(\mathbf{0}, \mathbf{\Sigma}_1),$$

$$\mathbf{\Sigma}_1 = \begin{bmatrix} 1.0, & 0.5 \\ 0.5, & 1.25 \end{bmatrix}$$

and the first order three-dimensional autoregressive process of the form

$$\mathbf{Y}(t) + \begin{bmatrix} -0.3, & -0.8, & -0.5 \\ 0.6, & -0.9, & 0.0 \\ -0.6, & 0.2, & -0.7 \end{bmatrix} \mathbf{Y}(t-1) = \boldsymbol{\psi}_2(t),$$

where

$$\boldsymbol{\psi}_2(t) \sim N(\mathbf{0}, \mathbf{\Sigma}_2),$$

$$\mathbf{\Sigma}_2 = \begin{bmatrix} 1.0, & 0.5, & -0.4 \\ 0.5, & 1.25, & -0.1 \\ -0.4, & -0.1, & 1.17 \end{bmatrix},$$

$$E[\boldsymbol{\psi}_1(r)\,\boldsymbol{\psi}_2'(s)] = \begin{cases} \mathbf{\Sigma}_{12}\,, & \text{for } r = s, \\ \mathbf{0}\,, & \text{for } r \neq s, \end{cases}$$

$$\mathbf{\Sigma}_{12}=\begin{bmatrix}1.25\,, & 1.125\,, & -1.4\\ 1.125, & 1.8125, & -0.7\end{bmatrix}.$$

We shall express the covariance matrices $\mathbf{C}_\mathbf{X}(0)$, $\mathbf{C}_\mathbf{Y}(0)$ and cross-covariance matrix $\mathbf{C}_\mathbf{XY}(0)$ by the parameters of the autoregressive processes $\mathbf{X}(t)$ and $\mathbf{Y}(t)$.

The matrix **A** associated with the process $\mathbf{X}(t)$ has the form

$$\mathbf{A}=\begin{bmatrix}1.0, & 0.3, & 0.02, & 0.0\\ -3.3, & -1.0, & 0.0\,, & 0.02\\ 1.0, & 0.0, & 0.0\,, & 0.0\\ 0.0, & 1.0, & 0.0\,, & 0.0\end{bmatrix}.$$

The eigenvalues of this matrix are $\lambda_{11}=-0.2$, $\lambda_{12}=0.2$, $\lambda_{13}=0.1$, $\lambda_{14}=-0.1$ and the matrix whose columns are the associated eigenvectors has the form

$$\mathbf{P}_1=\begin{bmatrix}0.25 & -0.55 & -2.99 & -6.18\\ -0.92 & 1.65 & 10.96 & 18.54\\ -1.26, & -2.75 & -29.89 & 61.79\\ 4.62 & 8.25 & 109.60 & -185.38\end{bmatrix}.$$

The matrix $\mathbf{W}_1$ has the form

$$\mathbf{W}_1=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix}.$$

The matrix **B** associated with the process $\mathbf{Y}(t)$ has the form

$$\mathbf{B}=\begin{bmatrix}0.3 & 0.8 & 0.5\\ -0.6 & 0.9 & 0.0\\ 0.6 & -0.2 & 0.7\end{bmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_{21}=0.6+0.3i,\quad \lambda_{22}=0.6-0.3i,\quad \lambda_{23}=0.7$$

and the matrix whose columns are the associated eigenvectors has the form

$$\mathbf{P}_2=\begin{bmatrix}0.5+0.5i & 0.5-0.5i & -1\\ i & -i & -3\\ -i & i & 4\end{bmatrix}.$$

The matrix $\mathbf{W}_2$ has the form

$$\mathbf{W}_2=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

The covariance matrices and the cross-covariance matrix of the processes $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ have the form

$$\mathbf{C}_\mathbf{X}(0)=\begin{bmatrix}2.42 & -4.18\\ -4.18 & 16.73\end{bmatrix},$$

$$C_Y(0) = \begin{bmatrix} 4.17 & 1.33 & 0.88 \\ 1.33 & 6.89 & -6.29 \\ 0.88 & -6.29 & 10.05 \end{bmatrix},$$

$$C_{XY}(0) = \begin{bmatrix} 2.27 & 1.65 & -1.89 \\ -2.2 & 0.06 & 0.93 \end{bmatrix}.$$

The canonical correlations are

$$\rho_1 = 0.9702, \quad \rho_2 = 0.2382$$

and the canonical series are

$$U_1(t) = 0.7998X_1(t) + 0.1151X_2(t),$$
$$U_2(t) = 0.2954X_1(t) + 0.3031X_2(t),$$
$$V_1(t) = 0.4985Y_1(t) - 0.1613Y_2(t) - 0.2886Y_3(t),$$
$$V_2(t) = -0.2668Y_1(t) + 0.6427Y_2(t) + 0.3102Y_3(t).$$

The coefficients of determination are

$\mathrm{corr}^2(X_i(t), V_j(t)) \cdot 100$	$U_1(t)$	$U_2(t)$
$X_1(t)$	87.39	12.61
$X_2(t)$	12.00	88.00

$\mathrm{corr}^2(Y_i(t), V_j(t)) \cdot 100$	$V_1(t)$	$V_2(t)$	$V_3(t)$
$Y_1(t)$	62.18	0.01	37.81
$Y_2(t)$	27.13	65.36	7.51
$Y_3(t)$	20.85	13.39	65.76

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